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Nondegeneracy and Integral Representation of Weak Markov Systems

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We generalize a result of Schwenker concerning the integral representation of weak Markov systems. © 1992 Academic Press, Inc.

In the sequel, A will denote a subset of the real line containing at least $n+2$ elements, and $I(A)$ will denote the convex hull of A. A is said to satisfy *Property B* if between any two distinct points of A is another point of A.

A sequence of functions $Z_n := (z_0, ..., z_n)$ defined on A is called a (weak) *Tchebycheff system* if it is linearly independent and for all points $x_0 < \cdots < x_n$ in A, $\det\{z_i(x_j)\}_{i,j=0}^{n} > 0$ ($\geqslant 0$). If Z_k is a (weak) Tchebycheff system for $k = 0, ..., n$, we say that Z_n is a (weak) *Markov system.* Note that, in this case, $z_0 > 0$ ($z_0 \geqslant 0$). If $z_0 \equiv 1$ we say that Z_n is *normalized.* In the following definitions, when we say that a basis $U_n = \{u_0, ..., u_n\}$ is obtained from Z_n by a triangular linear transformation, we mean that $u_0 \equiv z_0$ and $u_k - z_k \in S(Z_{k-1})$ ($k = 1, ..., n$), where $S(Z_k)$ denotes the linear span of Z_k.

DEFINITION 1. Z_n is said to be *endpoint nondegenerate* (END) provided that for every c in A, the restrictions of $S(Z_n)$ to $A \cap (-\infty, c)$ and to $A \cap (c, \infty)$ have the same dimension as $S(Z_n)$. This term, coined by D. J. Newman, was first used by Zwick in [11]. It was also used by Zielke in [10], where it is referred to simply as "nondegeneracy."

DEFINITION 2. Z_n is said to satisfy *Condition E* if for all $c \in I(A)$ the following two requirements are satisfied:

(a) If Z_n is linearly independent on $[c, \infty) \cap A$ then there exists a basis $(u_0, ..., u_n)$ for $S(Z_n)$, obtained by a triangular linear transformation,

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such that for any sequence of integers $0 \leqslant k(0) < \cdots < k(m) \leqslant n$, $(u_{k(r)})_{r=0}^{m}$ is a weak Markov system on $A \cap [c, \infty)$.

(b) If Z_n is linearly independent on $(-\infty, c)] \cap A$ then there exists a basis $(v_0, ..., v_n)$ for $S(Z_n)$, obtained by a triangular linear transformation, such that for any sequence of integers $0 \leqslant k(0) < \cdots < k(m) \leqslant n$, $((-1)^{r-k(r)} v_{k(r)})_{r=0}^{m}$ is a weak Markov system on $(-\infty, c] \cap A$.

Definition 3. Z_n is said to satisfy *Condition I* if for every real number c, Z_n is linearly independent on at least one of the sets $(-\infty, c) \cap A$ and $A \cap (c, \infty)$.

Definition 4. Z_n is called *weakly nondegenerate* if it satisfies both of conditions I and E.

Definition 5. Z_n is *representable* if and only if, for all $c \in A$ there is a basis U_n, obtained from Z_n by a triangular linear transformation (hence, $u_0(x) = z_0(x)$), a strictly increasing function h (an "embedding function") defined on A, with $h(c) = c$, and a sequence $W_n = (w_1, ..., w_n)$ of continuous, increasing functions defined on $I(h(A))$, such that

$$u_1(x) = u_0(x) \int_c^{h(x)} dw_1(t_1)$$

$$\vdots$$

$$u_n(x) = u_0(x) \int_c^{h(x)} \int_c^{t_1} \cdots \int_c^{t_{n-1}} dw_n(t_n) \cdots dw_1(t_1).$$

In this case we will say that (h, c, W_n, U_n) is a *representation* of Z_n.

Although there have been earlier attempts [2, 5], the first correct result linking weak Markov systems and representability is due to Zielke [10], who proved that an END weak Markov system is representable (the representability of some classes of Markov systems has been known for a long time; see, e.g., [1]). Zielke's result was generalized by this author in [7], where it was shown that if Z_n is a weakly nondegenerate normalized weak Markov system, then it is representable. This result was in turn improved by Schwenker [3], who proved that if Z_n is a normalized weak Markov system, then Condition E is satisfied if and only if Z_n is representable. In this paper we show how Schwenker's result can be obtained by making slight changes in the arguments developed in [7]. Using a result proved in [8], we also obtain a generalization of Schwenker's theorem. Before continuing we must introduce an additional definition.

DEFINITION 6. Let $W_n = (w_1, ..., w_n)$ be a sequence of real-valued functions defined on (a, b), let h be a real-valued function defined on A with $h(A) \subset (a, b)$, and let $x_0 < \cdots < x_n$ be points of $h(A)$. We say that W_n satisfies *Property M* with respect to h at $(x_0, ..., x_n)$ if there is a sequence $(t_{i,j}: i = 0, ..., n; j = 0, ..., n-i)$ in $h(A)$ such that

(a) $x_j = t_{0,j}$ $(j = 0, ..., n)$;

(b) $t_{i,j} < t_{i+1,j} < t_{i,j+1}$ $(i = 0, ..., n-1; j = 0, ..., n-i)$.

(c) For $i = 1, ..., n$, $w_i(x)$ is not constant at $t_{i,j}$ $(j = 0, ..., n-i)$.

To say that a function f is not constant at a point $c \in (a, b)$ is to say that for every $\varepsilon > 0$ there are points $x_1, x_2 \in (a, b)$ with $c - \varepsilon < x_1 < c < x_2 < c + \varepsilon$, such that $f(x_1) \neq f(x_2)$.

THEOREM 1. *Let Z_n be defined on a set A. Then the following statements are equivalent:*

(a) *Z_n is a normalized weak Markov system that satisfies Condition E.*

(b) *Z_n is representable, and there is a representation (h, c, W_n, U_n) of Z_n such that W_n satisfies Property M with respect to h at some sequence $x_0 < \cdots < x_n$ in A.*

(c) *Z_n is representable, and for every representation (h, c, W_n, U_n) of Z_n, W_n satisfies Property M with respect to h at some sequence $x_0 < \cdots < x_n$ in A.*

DEFINITION 7. A function f defined on an open interval I is said to be c-absolutely continuous if it is absolutely continuous in every compact subset of I.

DEFINITION 8. Let $Z_n := (z_0, ..., z_n)$ be a sequence of functions defined on a set A, and let $V_n := (v_0, ..., v_n)$ be a sequence of functions defined on a set B. We say that Z_n can be embedded in V_n if there is a strictly increasing function $h: A \to B$ such that $v_i[h(t)] = z_i(t)$ for every $t \in A$ and $i = 0, ..., n$. The function h is called an *embedding function.*

In the proof of Theorem 1 we shall need the following analog of [7, Theorem 3]:

THEOREM 2. *Let $c \in A$. If Z_n is a normalized weak Markov system on A that satisfies Condition E, then it can be embedded in a normalized weak Markov system V_n of c-absolutely continuous functions defined on an open interval and satisfying condition E there, and V_n and the embedding function*

$h(t)$ can be chosen so that $h(c)=c$. Moreover, if A satisfies Property B, the converse statement is also true.

The proof of Theorem 2 is based on the following auxiliary propositions.

LEMMA 1. *Let Z_n be a weak Markov system on a set A, satisfying Condition E, let $p: A \to R$ be a strictly increasing function, and let $v_r(t) := z_r(p^{-1}(t))$, $r=0, ..., n$. Then $V_n := (v_0, ..., v_n)$ is a weak Markov system on $p(A)$ that satisfies Condition E.*

The proof of Lemma 1 is straightforward and will be omitted.

LEMMA 2. *Let $[a, b]$ be a compact interval, $f \in C[a, b]$ of bounded variation and $g \in C[a, b]$ strictly increasing. For $a \leqslant \alpha \leqslant \beta \leqslant b$, let $V(f, \alpha, \beta)$ denote the total variation of f on $[\alpha, \beta]$. Let $c \in [a, b]$ be arbitrarily fixed, and define $v(f, t)$ to equal $V(f, c, t)$ on $[c, b]$ and $-V(f, t, c)$ on $[a, c)$. Finally, let $q(t)=g(t)+v(f, t)$ and $h(t)=f[q^{-1}(t)]$. Then $h(t)$ is absolutely continuous on $[q(a), q(b)]$.*

This is [7, Lemma 2].

LEMMA 3. *Let $U_n := (u_0, ..., u_n)$ be a normalized weak Markov system on a set A, satisfying Condition E, let $l_1 := \inf(A)$, $l_2 := \sup(A)$, $c \in I(A)$, and assume that $u \in S(U_n)$.*

(a) *If $c > l_1$ and c is a point of accumulation of $(l_1, c) \cap A$, then $\lim_{t \to c^-} u(t)$ exists and is finite.*

(b) *If $c < l_2$ and c is a point of accumulation of $(c, l_2) \cap A$, then $\lim_{t \to c^+} u(t)$ exists and is finite.*

Proof. We only prove (a); the proof of (b) is similar and will be omitted.

We proceed by induction. The assertion is trivially true for $n=0$. To prove the inductive step, assume that for any function w in $S(U_{n-1})$ (where $U_{n-1}=(u_0, ..., u_{n-1})$) $\lim_{t \to c^-} w(t)$ exists and is finite. Since the definition of a weak Markov system implies that U_n is linearly independent, it is easy to see that there is a point $d \in A \cap [c, \infty)$ such that U_n is linearly independent in $(-\infty, d] \cap A$. From Condition E we conclude that there is a function $u=u_n+w$, with $w \in S(U_{n-1})$, such that u is monotonic on $(-\infty, d] \cap A$; whence the conclusion readily follows. Q.E.D.

LEMMA 4. *Let Z_n be a normalized weak Markov system of bounded functions defined on a compact interval $I=[a, b]$. Then all the elements of $S(Z_n)$ are of bounded variation on I.*

This is [7, Lemma 4].

LEMMA 5. *Let Z_n be a normalized weak Markov system on an interval I (open, closed, or semiopen) that satisfies Condition E, and let $c \in I$. If z_1 is continuous at c, then all the elements of $S(Z_n)$ are continuous at c.*

Proof. We shall only prove that if $c > \inf(I)$, then all the elements of $S(Z_n)$ are left-continuous at c. The proof of the other case is similar and will be omitted.

We proceed by induction on n. For $n = 1$ the assertion is true by hypothesis; let therefore $n > 1$. Assume first that z_1 is constant on $(\inf(I), c)$. The linear independence implies there is a $t_1 \in (c, \sup(I))$ such that $z_1(c) < z_1(t_1)$. As in the proof of Lemma 3 we deduce that there is a point $c_1 < c$ in I such that Z_n is linearly independent on $S_1 := [c_1, \infty) \cap I$. Thus from Condition E(a) we conclude that there is a sequence U_n, obtained from Z_n by a triangular linear transformation, such that both $(1, u_n)$ and $(1, u_1, u_n)$ are weak Markov systems on S_1. The first assertion is equivalent to saying that u_n is increasing on S_1, from which we conclude that $u_n(c^-) \leqslant u_n(c)$. Let $c_1 < t < c$. Then $u_1(t) = u_1(c) < u_1(t_1)$. If $u_1(c) = 0$ the next identity is obvious, whereas for $u_1(c) > 0$ it follows by subtracting $u_1(c)$ times the first row, from the second row:

$$0 \leqslant \begin{vmatrix} 1 & 1 & 1 \\ u_1(t) & u_1(c) & u_1(t_1) \\ u_n(t) & u_n(c) & u_n(t_1) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & u_1(t_1) - u_1(c) \\ u_n(t) & u_n(c) & u_n(t_1) \end{vmatrix}$$

$= -[u_1(t_1) - u_1(c)][u_n(c) - u_n(t)]$, whence $u_n(c) \leqslant u_n(t)$. Passing to the limit we infer that $u_n(c) \leqslant u_n(c^-)$, and therefore $u_n(c^-) = u_n(c)$.

Assume now that there is a $t_2 \in (\inf(I), c)$ such that $z_1(t_2) < z_1(c)$. There is a $c_2 \in (c, \sup(I))$ such that Z_n is linearly independent on $S_2 := (-\infty, c_2] \cap I$. From Condition E(b) we conclude that there is a sequence U_n, obtained from Z_n by a triangular linear transformation, such that both $(1, (-1)^{n-1} u_n)$ and $(1, u_1, (-1)^n u_n)$ are weak Markov systems on S_2. From the first assertion we infer that $(-1)^{n-1} u_n(c^-) \leqslant (-1)^{n-1} u_n(c)$. It is also clear that $u_1(t_2) < u_1(c)$. For the sake of completeness we reproduce the argument used in the proof of [7, Lemma 5]. If $t_2 < t < c$, we have

$$0 \leqslant \begin{vmatrix} 1 & 1 & 1 \\ u_1(t_2) & u_1(t) & u_1(c) \\ (-1)^n u_n(t_2) & (-1)^n u_n(t) & (-1)^n u_n(c) \end{vmatrix}$$

$$= (-1)^n \begin{vmatrix} 1 & 1 & 0 \\ u_1(t_2 & u_1(t) & u_1(c) - u_1(t) \\ u_n(t_2) & u_n(t) & u_n(c) - u_n(t) \end{vmatrix}.$$

Since $u_1(t)$ is continuous at c, passing to the limit we have

$$0 \leqslant (-1)^n \begin{vmatrix} 1 & 1 & 0 \\ u_1(t_2) & u_1(c) & 0 \\ u_n(t_2) & u_n(c^-) & u_n(c)-u_n(c^-) \end{vmatrix}$$
$$= (-1)^n [u_n(c)-u_1(t_2)][u_n(c)-u_n(c^-)];$$

whence we conclude that $(-1)^{n-1} u_n(c) \leqslant (-1)^{n-1} u_n(c^-)$.

We have thus far shown that $u_n(t)$ is left-continuous at c. Since $u_n = z_n + w$, with $w \in S(Z_{n-1})$, applying the inductive hypothesis we conclude that also z_n is left-continuous at c. Q.E.D.

Proof of Theorem 2. The proof is achieved by repeating "verbatim" the proof of [7, Theorem 3], but omitting that part of the argument that is associated with the proof of Condition I. Q.E.D.

To prove Theorem 1 we also need the following two lemmas:

LEMMA 6. *Let $U_n := (u_0, ..., u_n)$ be a weak Markov system on an interval (a, b), that satisfies Condition E. If for some $c \in (a, b)$, $u_0(c) = 0$, then $u_k(c) = 0$, $k = 1, 2, ..., n$.*

Proof. We proceed by induction on n. For $n = 0$ the assertion is true by hypothesis. To prove the inductive step note first that there is a point $t_0 \in (a, b)$ such that $u_0(t_0) \neq 0$. Assume, e.g., that $t_0 < c$. It is readily seen that there is a d, $c < d < b$, such that U_n is linearly independent on $(a, d]$. Then, from Condition E we infer that there is a sequence V_n, obtained from U_n by a triangular linear transformation, such that both $((-1)^n v_n)$ and $(v_0, (-1)^{n+1} v_n)$ are weak Markov systems on $(a, d]$. From the first condition we infer that $(-1)^n v_n(c) \geqslant 0$. Applying the second condition we have $v_0(t_0) \geqslant 0$ (since $v_0(t_0) \neq 0$, this implies that $v_0(t_0) > 0$), and

$$0 \leqslant \begin{vmatrix} v_0(t_0), & 0 \\ (-1)^{n+1} v_n(t_0), & (-1)^{n+1} v_n(c) \end{vmatrix} = v_0(t_0)(-1)^{n+1} v_n(c);$$

whence $(-1)^n v_n(c) \leqslant 0$. We thus conclude that $v_n(c) = 0$. Assume now that $t_0 > c$. It is readily seen that there is a d, $a < d < c$, such that U_n is linearly independent on $[d, b)$. Applying Condition E we conclude that there is a set V_n, obtained from U_n by a triangular linear transformation, such that both (v_n) and (v_0, v_n) are weak Markov systems on $[d, b)$. The first condition implies that $v_n(c) \geqslant 0$. The second condition implies that $v_0(t_0) > 0$ and

$$0 \leqslant \begin{vmatrix} 0, & v_0(t_0) \\ v_n(c), & v_n(t_0) \end{vmatrix} = -v_0(t_0)\, v_n(c);$$

whence $v_n(c) \leqslant 0$. Therefore, $v_n(c) = 0$. Q.E.D.

LEMMA 7. *Let* $W_n := (w_1, ..., w_n)$ *be a sequence of increasing and continuous functions defined on an open interval* (a, b), *let* $c \in (a, b)$, $u_0 \equiv 1$, *and let* $u_k(x) := \int_c^x \int_c^{t_1} \cdots \int_c^{t_{k-1}} dw_k(t_k) \cdots dw_1(t_1)$ *for* $k = 1, ..., n$. *Assume* $a < x_0 < \cdots < x_n < b$; *then* $\det[u_i(x_j); i, j = 0, ..., n] > 0$ *if and only if* W_n *satisfies Property M with respect to the identity function at* $(x_0, ..., x_n)$.

This is the Lemma of [8].

Proof of Theorem 1. a $\Rightarrow$ c. To prove that Z_n is representable we simply repeat "verbatim" the proof of [7, Theorem 1], but replacing every occurrence of the phrase "weakly nondegenerate" with "Condition E." One of the steps in this proof has been described too tersely, making the argument difficult to follow: On [7, p. 9], to show that Q'_{n-1} is a (weakly nondegenerate) weak Markov system on D, the reader is referred to a book by Zielke [9, Theorem 11.3(b)]. Zielke, however, has a more general definition of a weak Markov system, and so only proves the weak constancy of the sign of the corresponding determinant, whereas we need to prove that it is actually nonnegative. To do this, assume that for some k and every $t_1 < \cdots < t_k$ in D, $\det(q_i'(t_j))_{i,j=1}^k \leqslant 0$. Since $q_i(s_j) - q_i(s_{j-1}) = \int_{s_{j-1}}^{s_j} q_i'(t)\, dt$, proceeding as in, e.g., the proof of [1, p. 382, Lemma 1], we conclude that for every $s_0 < \cdots < s_k$ in (a_1, b_1) (where (a_1, b_1) is defined in [7, p. 8]), $\det(q_i(s_j))_{i,j=0}^k \leqslant 0$, a contradiction.

Another, and simpler, way of showing that Q'_{n-1} is a weak Markov system on D, consists in applying [4, Lemma 1].

Note that in [7, p. 9], in the paragraph that begins on line 11, there is no need to use Condition I, since the linear independence of Q'_{n-1} on $(a_1, b_2] \cap D$ follows from the linear independence of Q'_{n-1} on D. Let (h, c, W_n, U_n) be a representation of Z_n. Since Z_n is linearly independent by definition, so is U_n, and the conclusion readily follows from Lemma 7.

c $\Rightarrow$ b. Trivial.

b $\Rightarrow$ a. Since there is a representation (h, c, W_n, U_n) such that W_n satisfies Property M with respect to h at some sequence $x_0 < \cdots < x_n$ in A, applying Lemma 7 we deduce that U_n (and hence Z_n) is linearly independent. The assertion that U_n (and hence Z_n) is a weak Markov system is proved by a procedure similar to that employed in, e.g., the proof of [1, p. 382, Lemma 1]. The proof of Condition E also uses a similar argument (see [6, p. 205] for more details). Q.E.D.

REFERENCES

1. S. KARLIN AND W. J. STUDDEN, "Tchebycheff Systems: With Applications in Analysis and Statistics," Interscience, New York, 1966.

2. M. A. RUTMAN, Integral representation of functions forming a Markov series, *Soviet Math. Dokl.* **164** (1965), 1340–1343.
3. F. SCHWENKER, "Integraldarstellung schwacher Markov-Systeme," dissertation, University of Osnabrück, Sept., 1988.
4. R. A. ZALIK, Smoothness properties of generalized convex functions, *Proc. Amer. Math. Soc.* **56** (1976), 118–120.
5. R. A. ZALIK, Integral representation of Tchebycheff systems, *Pacific J. Math.* **68** (1977), 553–568.
6. R. A. ZALIK AND D. ZWICK, On extending the domain of definition of Čeybšev and weak Čebyšev systems, *J. Approx. Theory* **57** (1989), 202–210.
7. R. A. ZALIK, Integral representation and embedding of weak Markov systems, *J. Approx. Theory* **58** (1989), 1–11.
8. R. A. ZALIK, Integral representation of Markov systems and the existence of adjoined functions for Haar spaces, *J. Approx. Theory* **65** (1991), 22–31.
9. R. ZIELKE, "Discontinuous Čebyšev Systems," Lecture Notes in Mathematics, Vol. 707, Springer-Verlag, Berlin/New York, 1979.
10. R. ZIELKE, Relative differentiability and integral representation of a class of weak Markov systems, *J. Approx. Theory* **44** (1985), 30–42.
11. D. ZWICK, Some hereditary properties of WT-systems, *J. Approx. Theory* **41** (1984), 114–134.